

# A DIFFERENCE ANALOGUE OF SECOND MAIN THEOREM FOR HOLOMORPHIC CURVE INTO ALGEBRAIC VARIETY

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**ABSTRACT.** In this paper, we prove the difference analogue of second main theorem for holomorphic curves on  $\mathbb{C}$  intersecting a finite set of fixed hypersurfaces in general position on algebraic variety  $V$  which is a difference counterpart of the second main theorem due to M. Ru [10]. As an application, we prove a result for algebraically degenerate of holomorphic curve on  $\mathcal{P}_c^1$  intersecting hypersurfaces.

## 1. INTRODUCTION AND MAIN RESULTS

Recently, the second main theorem of Nevanlinna have studied for difference operator. In 2006, R. Halburd and R. Korhonen [6, 7] have built the second main theorem for difference operator of meromorphic functions. In 2014, R. Korhonen et. al [5] have proved the difference analogue of second main theorem for holomorphic curves on  $\mathbb{C}$  intersecting a finite set of fixed hyperplane in general position on  $\mathbb{P}^n(\mathbb{C})$ . In this paper, our idea is to establish the difference analogue of second main theorem for holomorphic curves on  $\mathbb{C}$  intersecting a finite set of fixed hypersurfaces in general position on algebraic variety  $V$ .

Let  $f$  be a holomorphic curve of  $\mathbb{C}$  into  $\mathbb{P}^n(\mathbb{C})$ . For arbitrary fixed homogeneous coordinates  $(w_0 : \cdots : w_n)$  of  $\mathbb{P}^n(\mathbb{C})$ , we take a reduced representation  $f = (f_0 : \cdots : f_n)$  which means that each  $f_i$  is a holomorphic function on  $\mathbb{C}$  without common zeros. Set  $\|f(z)\| = \max\{|f_0(z)|, \dots, |f_n(z)|\}$ , the characteristic function of  $f$  is defined by

$$T_f(r) = \frac{1}{2\pi} \int_0^{2\pi} \log \|f(re^{i\theta})\| d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of  $f$ .

We denote the *hyper-order* and *order* of holomorphic curve  $f$  by

$$\varsigma(f) = \limsup_{r \rightarrow \infty} \frac{\log \log T_f(r)}{\log r} \text{ and } \sigma(f) = \limsup_{r \rightarrow \infty} \frac{\log T_f(r)}{\log r}, \text{ respectively.}$$

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Let  $D$  be a hypersurface in  $\mathbb{P}^n(\mathbb{C})$  of degree  $d$ . Let  $Q$  be the homogeneous polynomial of degree  $d$  defining  $D$ . Under the assumption that  $Q(f) \not\equiv 0$ . Then, the proximity function  $m_f(r, D)$  of  $f$  is defined by

$$m_f(r, D) = \frac{1}{2\pi} \int_0^{2\pi} \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} d\theta,$$

where the above definition is independent, up to an additive constant, of the choice of the reduced representation of  $f$ .

The definition as following is given by R. Korhonen et. al in [8].

**Definition 1.** Let  $n \in \mathbb{N}^*$ ,  $c \in \mathbb{C} \setminus \{0\}$  and  $a \in \mathbb{P}^1(\mathbb{C})$ . An  $a$ -point  $z_0$  of a meromorphic function  $h(z)$  is said to be  $n$ -successive and  $c$ -separated, if the  $n$  meromorphic functions  $h(z + jc)$  ( $j = 1, \dots, n$ ) take the value  $a$  at  $z = z_0$  with multiplicity not less than that of  $h(z)$  there. All the other  $a$ -points of  $h(z)$  are called  $n$ -aperiodic of pace  $c$ . By  $\tilde{N}_h^{[n,c]}(r, a)$  we denote the counting function of  $n$ -aperiodic zeros of the function  $h - a$  of pace  $c$ .

Therefore we denote  $\tilde{N}_g^{[n,c]}(r, D)$  the counting function of  $n$ -aperiodic zeros of the function  $Q(g)$  for holomorphic curve  $g : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$ . This means

$$\tilde{N}_g^{[n,c]}(r, D) = \tilde{N}_{Q(g)}^{[n,c]}(r, 0).$$

In order to state our result, we need some definitions as following.

Let  $V \subset \mathbb{P}^N(\mathbb{C})$  be a smooth complex projective variety of dimension  $n \geq 1$ . Let  $D_1, \dots, D_q$  be hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$ , where  $q > n$ . The hypersurfaces  $D_1, \dots, D_q$  are said to be *in general position on  $V$*  if for every subset  $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$ , we have

$$V \cap \text{Supp} D_{i_0} \cap \dots \cap \text{Supp} D_{i_n} = \emptyset,$$

where  $\text{Supp}(D)$  means the support of the divisor  $D$ .

Let  $c \in \mathbb{C} \setminus \{0\}$ , and  $\mathcal{P}_c^1$  be the field of period  $c$  meromorphic functions in  $\mathbb{C}$  of hyper-order strictly less than one. Let  $f : \mathbb{C} \rightarrow V$  be holomorphic curve. Suppose that  $V$  is generated by the homogeneous ideal  $\mathcal{I}(V)$ . We denote  $\mathcal{I}_{\mathcal{P}_c^1}(V)$  by the ideal in  $\mathcal{P}_c^1[x_0, \dots, x_N]$  generated by  $\mathcal{I}(V)$ . Let  $\mathcal{V}$  is a subset algebraic (zero locus) which is generated by the ideal  $\mathcal{I}_{\mathcal{P}_c^1}(V)$  on  $\mathcal{P}_c^1[x_0, \dots, x_N]$ . We say that  $f$  is algebraically nondegenerate over  $\mathcal{P}_c^1$  if the image of  $f$  is not contained in any proper subset algebraic of

$$\mathcal{V} = Z(\mathcal{I}_{\mathcal{P}_c^1}(V)).$$

Our result is the following theorem:

**Theorem 1.** *Let  $V \subset \mathbb{P}^N(\mathbb{C})$  be a complex projective variety of dimension  $n \geq 1$ . Let  $D_1, \dots, D_q$  be hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$  of degree  $d_j$ , located in general position on  $V$ . Let  $d$  be the least common multiple of the  $d_i, i = 1, \dots, q$ . Let  $f = (f_0 : f_1 : \dots : f_N) : \mathbb{C} \rightarrow V$  be an algebraically non-degenerate holomorphic map on  $\mathcal{P}_c^1$  with  $\varsigma(f) = \varsigma < 1$ . Let  $\varepsilon > 0$ , then we have for any  $1 < r < +\infty$ ,*

$$\| \quad (q(1 - \varepsilon/3) - (n + 1) - \varepsilon/3)T_f(r) \leq \sum_{l=1}^q d_l^{-1} \tilde{N}_f^{[M, c]}(r, D_l) + S(r, f).$$

*outside of a possible exceptional set  $E \subset [1, \infty)$  of finite logarithmic measure, where  $M = \frac{n^n d^{n^2+n} (19nI(\varepsilon^{-1}))^n \cdot (\deg V)^{n+1}}{n!}$ ,  $I(x) := \min\{k \in \mathbb{N} : k > x\}$  for each positive constant  $x$ .*

From Theorem 1, taking  $\varepsilon = \frac{3(1-K)}{2(n+3)}$ , where  $K \in [0, a], 0 < a < 1$ , we obtain the following result:

**Corollary 1.** *Let  $V \subset \mathbb{P}^N(\mathbb{C})$  be a complex projective variety of dimension  $n \geq 1$ . Let  $D_1, \dots, D_{n+2}$  be hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$  of degree  $d_j$ , located in general position on  $V$ . Let  $f = (f_0 : f_1 : \dots : f_N) : \mathbb{C} \rightarrow V$  be a holomorphic curve with  $\varsigma(f) = \varsigma < 1$ . Suppose that*

$$\sum_{l=1}^{n+2} d_l^{-1} \tilde{N}_f^{[M, c]}(r, D_l) \leq K T_f(r) + S(r, f),$$

*then  $f$  is a algebraically degenerate holomorphic map on  $\mathcal{P}_c^1$ .*

Corollary 1 is the first result for algebraically degenerate of holomorphic curve on  $\mathcal{P}_c^1$  intersecting hypersurfaces.

Taking  $N = n$  in Theorem 1, we get the following result.

**Corollary 2.** *Let  $D_1, \dots, D_q$  be hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$  of degree  $d_j$ , located in general position on  $\mathbb{P}^N(\mathbb{C})$ . Let  $d$  be the least common multiple of the  $d_i, i = 1, \dots, q$ . Let  $f = (f_0 : f_1 : \dots : f_N) : \mathbb{C} \rightarrow \mathbb{P}^N(\mathbb{C})$  be an algebraically non-degenerate holomorphic map on  $\mathcal{P}_c^1$  with  $\varsigma(f) = \varsigma < 1$ . Let  $\varepsilon > 0$ , then we have for any  $1 < r < +\infty$ ,*

$$\| \quad (q(1 - \varepsilon/3) - (N + 1) - \varepsilon/3)T_f(r) \leq \sum_{l=1}^q d_l^{-1} \tilde{N}_f^{[M, c]}(r, D_l) + S(r, f).$$

*outside of a possible exceptional set  $E \subset [1, \infty)$  of finite logarithmic measure, where  $M = \frac{N^N d^{N^2+N} (19NI(\varepsilon^{-1}))^N}{N!}$ ,  $I(x) := \min\{k \in \mathbb{N} : k > x\}$  for each positive constant  $x$ .*

## 2. SOME LEMMAS

In order to prove theorems, we need the following lemmas.

**Lemma 1.** [5] *Let  $f$  be a non-constant meromorphic function in  $\mathbb{C}$ , and let  $c \in \mathbb{C} \setminus \{0\}$ . If  $\varsigma(f) = \varsigma < 1$ , then*

$$m(r, \frac{f(z+c)}{f(z)}) = S(r, f),$$

*for all  $r > 0$  outside of a possible exceptional set  $E \subset [1, +\infty)$  of finite logarithmic measure  $\int_E dt/t < +\infty$ .*

**Lemma 2.** [5] *If the holomorphic curve  $g = [g_0 : \dots : g_n]$  satisfies  $\varsigma(g) < 1$  and  $c \in \mathbb{C} \setminus \{0\}$ , then  $C(g_0, \dots, g_n) \equiv 0$  if and only if the entires  $g_0, \dots, g_n$  are linearly dependent over the field  $\mathcal{P}_c^1$ .*

Let  $X \subset \mathbb{P}^N(\mathbb{C})$  be a projective variety of dimension  $n$  and degree  $\Delta$ . Let  $I_X$  be the prime ideal in  $\mathbb{C}[x_0, \dots, x_N]$  defining  $X$ . Denote by  $\mathbb{C}[x_0, \dots, x_N]_m$  the vector space of homogeneous polynomials in  $\mathbb{C}[x_0, \dots, x_N]$  of degree  $m$  (including 0). Put  $I_X(m) := \mathbb{C}[x_0, \dots, x_N]_m \cap I_X$ . The Hilbert function  $H_X$  of  $X$  is defined by

$$H_X(m) := \dim \mathbb{C}[x_0, \dots, x_N]_m / I_X(m).$$

For each tuple  $c = (c_0, \dots, c_N) \in \mathbb{R}_{\geq 0}^{N+1}$  and  $m \in \mathbb{N}$ , we define the  $m$ -th Hilbert weight  $S_X(m, c)$  of  $X$  with respect to  $c$  by

$$S_X(m, c) := \max \sum_{i=1}^{H_X(m)} I_i \cdot c_i,$$

where  $I_i = (I_{i0}, \dots, I_{iN}) \in \mathbb{N}_0^{N+1}$  and the maximum is taken over all sets  $\{x^{I_i} = x_0^{I_{i0}} \dots x_N^{I_{iN}}\}$  whose residue classes modulo  $I_X(m)$  form a basis of the vector space  $\mathbb{C}[x_0, \dots, x_N]_m / I_X(m)$ .

**Lemma 3.** [10] *Let  $X \subset \mathbb{P}^N(\mathbb{C})$  be an algebraic variety of dimension  $n$  and degree  $\Delta$ . Let  $m > \Delta$  be an integer and let  $c = (c_0, \dots, c_N) \in \mathbb{R}_{\geq 0}^{N+1}$ . Then*

$$\frac{1}{m H_X(m)} S_X(m, c) \geq \frac{1}{(n+1)\Delta} e_X(c) - \frac{(2n+1)\Delta}{m} \max_{0 \leq i \leq N} c_i.$$

**Lemma 4.** [10] *Let  $Y$  be a subvariety of  $\mathbb{P}^{q-1}(\mathbb{C})$  of dimension  $n$  and degree  $\Delta$ . Let  $c = (c_1, \dots, c_q)$  be a tupe of positive reals. Let  $\{i_0, \dots, i_n\}$  be a subset of  $\{1, \dots, q\}$  such that  $\{y_{i_0} = \dots = y_{i_n} = 0\} \cap Y = \emptyset$ . Then*

$$e_Y(c) \geq (c_{i_0} + \dots c_{i_n})\Delta.$$

For each holomorphic curve  $f : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$ , the Casorati determinant of  $f$  is defined by

$$C(f) = C(f_0, \dots, f_n) = \begin{vmatrix} f_0(z) & f_1(z) & \cdots & f_n(z) \\ f_0(z+c) & f_1(z+c) & \cdots & f_n(z+c) \\ \vdots & \vdots & \ddots & \vdots \\ f_0(z+nc) & f_1(z+nc) & \cdots & f_n(z+nc) \end{vmatrix},$$

where  $c \in \mathbb{C} \setminus \{0\}$ . Apply to Lemma 1, Lemma 2 and using method in [3], we get the result as follows.

**Lemma 5.** *Let  $f = (f_0 : f_1 : \cdots : f_n) : \mathbb{C} \rightarrow \mathbb{P}^n(\mathbb{C})$  be a linearly non-degenerate holomorphic curve on  $\mathcal{P}_c^1$  with  $\varsigma(f) < 1$ . Let  $H_1, \dots, H_q$  be hyperplanes in  $\mathbb{P}^n(\mathbb{C})$  in general position. Then we have the inequality*

$$\int_0^{2\pi} \max_K \sum_{l \in K} \log \frac{\|f(re^{i\theta})\|}{|(a_l, f)(re^{i\theta})|} \frac{d\theta}{2\pi} \leq (n+1)T_f(r) - N_{C(f)}(r, 0) + S(r, f).$$

holds for all  $r$  outside of an exceptional set of finite logarithmic measure. Here the maximum is taken over all subsets  $K$  of  $\{1, \dots, q\}$  such that  $a_l$ ,  $l \in K$ , are linearly independent.

### 3. PROOF OF THEOREM 1

*Proof.* Now, we are ready to prove the Theorem 1. Let  $D_1, \dots, D_q$  be the hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$  which are located in general position on  $V$ . Let  $Q_l$ ,  $1 \leq l \leq q$  be the homogeneous polynomials in  $\mathbb{C}[x_0, \dots, x_N]$  of degree  $d_l$  defining on  $D_l$ . We can replace  $Q_l$  by  $Q_l^{d/d_l}$ , where  $d$  is the l.c.m of  $d_l$ ,  $l = 1, \dots, q$ , we may assume that  $Q_1, \dots, Q_q$  have the same degree of  $d$ .

Given  $z \in \mathbb{C}$  there exists a renumbering  $\{i_0, \dots, i_n\}$  of the indices  $\{1, \dots, q\}$  such that

$$(3.1) \quad 0 < |Q_{i_0} \circ (f(z))| \leq |Q_{i_1} \circ (f(z))| \leq \cdots \leq |Q_{i_n} \circ (f(z))| \leq \min_{l \notin \{i_0, \dots, i_n\}} |Q_l \circ (f(z))|.$$

Let  $P_1, \dots, P_s$  be a base of algebraic variety  $V$ . From the hypothesis  $D_1, \dots, D_q$  be the hypersurfaces in  $\mathbb{P}^N(\mathbb{C})$  which are located in general position on  $V$ , we have for every subset  $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$ ,

$$V \cap \text{Supp} D_{i_0} \cap \cdots \cap \text{Supp} D_{i_n} = \emptyset.$$

This implies

$$P_1 \cap \cdots \cap P_s \cap \text{Supp} D_{i_0} \cap \cdots \cap \text{Supp} D_{i_n} = \emptyset.$$

Thus by Hilberts Nullstellensatz [11], for any integer  $k$ ,  $0 \leq k \leq N$ , there is an integer  $m_k > \{d, \max_{t=1}^s \{\deg P_t\}\}$  such that

$$x_k^{m_k} = \sum_{j=0}^n b_{k_j}(x_0, \dots, x_N) Q_{i_j}(x_0, \dots, x_N) + \sum_{t=1}^s b_t(x_0, \dots, x_N) P_t(x_0, \dots, x_N),$$

where  $b_{k_j}$  are homogeneous forms with coefficients in  $\mathbb{C}$  of degree  $m_k - d$  and  $b_t$  are homogeneous forms with coefficients in  $\mathbb{C}$  of degree  $m_k - \deg P_t$ ,  $t = 1, \dots, s$ . So from  $f : \mathbb{C} \rightarrow V$ , we have

$$\sum_{t=1}^s b_t(f_0(z), \dots, f_N(z)) P_t(f_0(z), \dots, f_N(z)) = 0.$$

This implies

$$|f_k(z)|^{m_k} \leq c_1 \|f(z)\|^{m_k-d} \max\{|Q_{i_0} \circ (f(z))|, \dots, |Q_{i_n} \circ (f(z))|\},$$

where  $c_1$  is a positive constant depends only on the coefficients of  $b_{k_j}$ ,  $0 \leq j \leq n$ ,  $0 \leq k \leq N$ , thus depends only on the coefficients of  $Q_l$ ,  $1 \leq l \leq q$ . Therefore,

$$(3.2) \quad \|f(z)\|^d \leq c_1 \max\{|Q_{i_0} \circ (f(z))|, \dots, |Q_{i_n} \circ (f(z))|\}.$$

By (3.1) and (3.2), we get

$$\prod_{l=1}^q \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f(z))|} \leq C \prod_{k=0}^n \frac{\|f(z)\|^d \|Q_{i_k}\|}{|Q_{i_k}(f(z))|},$$

where  $C = c_1^{q-n-1} \prod_{l \notin \{i_0, \dots, i_n\}} \|Q_l\|$  and  $\|Q_l\|$  is the maximum of the absolute values of the coefficients of  $Q_l$ . Thus, we have

$$\begin{aligned} \sum_{l=1}^q m_f(r, D_l) &= \int_0^{2\pi} \sum_{l=1}^q \log \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi} \\ &= \int_0^{2\pi} \log \prod_{l=1}^q \frac{\|f(re^{i\theta})\|^d}{|Q(f)(re^{i\theta})|} \frac{d\theta}{2\pi}. \end{aligned}$$

Hence, we get

$$(3.3) \quad \sum_{l=1}^q m_f(r, D_l) \leq \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \left\{ \log \prod_{k=0}^n \frac{\|f(re^{i\theta})\|^d}{|Q_{i_k}(f)(re^{i\theta})|} \right\} \frac{d\theta}{2\pi} + O(1).$$

By argument as M. Ru [10], we consider the map

$$\psi : x \in V \mapsto [Q_1(x) : \dots : Q_q(x)] \in \mathbb{P}^{q-1}(\mathbb{C}).$$

Taking  $Y = \psi(V)$ , the hypothesis *in general position* implies that  $\psi$  is a finite morphism on  $V$  and  $Y$  is a complex projective subvarieties of  $\mathbb{P}^{q-1}(\mathbb{C})$  and  $\dim Y = n$ ,  $\deg Y := \Delta \leq d^n \deg V$ . For each  $a = (a_1, \dots, a_q) \in \mathbb{Z}_{\geq 0}^q$ , we denote

by  $y^a = y_1^{a_1} \dots y_q^{a_q}$ . Let  $m$  be a positive integer, we consider the vector space  $V_m = \mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$ , where  $I_Y$  is the prime ideal which is defined algebraic variety  $Y$ ,  $(I_Y)_m := \mathbb{C}[y_1, \dots, y_q]_m \cap I_Y$ . Fix a basis  $\{\phi_0, \dots, \phi_{n_m}\}$  of  $V_m$ , where  $n_m + 1 = H_Y(m) = \dim V_m$ . Set,

$$F = [\phi_0(\psi \circ f) : \dots : \phi_{n_m}(\psi \circ f)] : \mathbb{C} \rightarrow \mathbb{P}^{n_m}(\mathbb{C}).$$

Note that,  $f$  is algebraically non-degenerate on  $\mathcal{P}_c^1$ , then  $F$  is linearly non-degenerate on  $\mathcal{P}_c^1$ . For any  $c \in \mathbb{R}_{\geq 0}^q$ , the Hilbert function of  $Y$  with respect to the weight  $c$  is defined by

$$S_Y(m, c) = \max \sum_{i=1}^{H_Y(m)} a_i \cdot c,$$

where the maximum is taken over all sets of monomials  $y^{a_1}, \dots, y^{a_{H_Y(m)}}$  whose  $y^{a_1} + (I_Y)_m, \dots, y^{a_{H_Y(m)}} + (I_Y)_m$  is a basis of  $\mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$ . For every  $z \in \mathbb{C}$ , let  $c_z = (c_{1,z}, \dots, c_{q,z})$ , where  $c_{l,z} = \log \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f(z))|}$ ,  $l = 1, \dots, q$ . We see that  $c_z \in \mathbb{R}_{\geq 0}^q$ , for all  $z \in \Omega$ . There exists a subset  $I_z \subset \{0, \dots, q_m\}$ ,  $q_m = C_{q+m-1}^m - 1$ ,  $|I_z| = n_m + 1 = H_Y(m)$  which  $\{y^{a_i} : i \in I_z\}$  is a basis of  $\mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$  (residue classes modulo  $(I_Y)_m$ ) and

$$S_Y(m, c_z) = \max \sum_{i=1}^{H_Y(m)} a_i \cdot c_z.$$

From two basis  $\{y^{a_i} : i \in I_z\}$  and  $\{\phi_0, \dots, \phi_{n_m}\}$ , there exist the forms independent linearly  $\{L_{l,z}, l \in I_z\}$  such that

$$y^{a_l} = L_{l,z}(\phi_0, \dots, \phi_{n_m}).$$

We denote  $J$  by the set of indices of the linear forms  $L_{l,z}$ . We see

$$\log \prod_{i \in J} \frac{1}{|L_i(F)(z)|} = \log \prod_{i \in J} \frac{1}{|Q_1(f)(z)|^{a_{i,1}} \dots |Q_q(f)(z)|^{a_{i,q}}}.$$

This implies

$$\begin{aligned} \max_J \log \prod_{i \in J} \frac{\|F(z)\|}{|L_i(F)(z)|} &\geq S_Y(m, c_z) - dm H_Y(m) \log \|f(z)\| \\ (3.4) \quad &+ (n_m + 1) \log \|F(z)\|. \end{aligned}$$

By Lemma 3, we have

$$(3.5) \quad S_Y(m, c_z) \geq \frac{m H_Y(m)}{(n+1)\Delta} e_Y(c_z) - H_Y(m)(2n+1)\Delta \cdot \max_{1 \leq i \leq q} c_{i,z}.$$

From Lemma 4 and  $D_1, \dots, D_q$  are in general position on  $V$ , for any  $\{i_0, \dots, i_n\} \subset \{1, \dots, q\}$ , we have

$$(3.6) \quad E_Y(c_z) \geq (c_{i_0,z} + \dots + c_{i_n,z})\Delta.$$

Using the definition of  $c_z$ , we obtain

$$(3.7) \quad c_{i_0,z} + \dots + c_{i_n,z} = \log \left( \frac{\|f(z)\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(z)|} \dots \frac{\|f(z)\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(z)|} \right).$$

From (3.4) to (3.7), we have

$$(3.8) \quad \begin{aligned} & \log \left( \frac{\|f(z)\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(z)|} \dots \frac{\|f(z)\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(z)|} \right) \\ & \leq \frac{(n+1)}{mH_Y(m)} \left( \max_J \log \prod_{l \in J} \frac{\|F(z)\|}{|L_l(F)(z)|} - (n_m + 1) \log \|F(z)\| \right) \\ & \quad + d(n+1) \log \|f(z)\| + \frac{(2n+1)(n+1)\Delta}{m} \max_{1 \leq i \leq q} c_{i,z} \\ & = \frac{(n+1)}{mH_Y(m)} \left( \max_J \log \prod_{l \in J} \frac{\|F(z)\|}{|L_l(F)(z)|} - (n_m + 1) \log \|F(z)\| \right) \\ & \quad + d(n+1) \log \|f(z)\| \\ & \quad + \frac{(2n+1)(n+1)\Delta}{m} \left( \max_{1 \leq l \leq q} \log \frac{\|f(z)\|^d \|Q_l\|}{|Q_l(f)(z)|} \right). \end{aligned}$$

Take the integration of both sides of (3.8), we have

$$\begin{aligned} & \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left( \frac{\|f(re^{i\theta})\|^d \|Q_{i_0}\|}{|Q_{i_0}(f)(re^{i\theta})|} \dots \frac{\|f(re^{i\theta})\|^d \|Q_{i_n}\|}{|Q_{i_n}(f)(re^{i\theta})|} \right) \frac{d\theta}{2\pi} \\ & \leq \frac{(n+1)}{mH_Y(m)} \left( \int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} - (n_m + 1) T_F(r) \right) \\ & \quad + d(n+1) T_f(r) + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) \end{aligned}$$

This implies

$$(3.9) \quad \begin{aligned} & \int_0^{2\pi} \max_{\{i_0, \dots, i_n\}} \log \left( \frac{\|f(re^{i\theta})\|^d}{|Q_{i_0}(f)(re^{i\theta})|} \dots \frac{\|f(re^{i\theta})\|^d}{|Q_{i_n}(f)(re^{i\theta})|} \right) \frac{d\theta}{2\pi} \\ & \leq \frac{(n+1)}{mH_Y(m)} \left( \int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} \right) \\ & \quad - \frac{(n+1)}{mH_Y(m)} (n_m + 1) T_F(r) + d(n+1) T_f(r) \\ & \quad + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) + O(1). \end{aligned}$$



Note that  $F$  is linearly non-degenerate on  $\mathcal{P}_c^1$ . Thus, apply to Lemma 5 for  $F$  and collection of hyperplanes  $L_l, l \in J$ , for every  $\varepsilon > 0$  and  $m$  is large enough, we obtain

$$(3.10) \quad \begin{aligned} & \parallel \frac{(n+1)}{mH_Y(m)} \left( \int_0^{2\pi} \max_J \log \prod_{l \in J} \frac{\|F(re^{i\theta})\|}{|L_l(F)(re^{i\theta})|} \frac{d\theta}{2\pi} - (n_m + 1)T_F(r) \right) \\ & \leq -\frac{n+1}{mH_Y(m)} N_{C(F)}(r, 0) + \frac{\varepsilon}{3m} T_F(r) + S(r, F), \end{aligned}$$

where  $N_{C(F)}(r, 0)$  is denoted by the counting function of Casorati determinant of  $F$ . Combining (3.3), (3.9) and (3.10), we get

$$(3.11) \quad \begin{aligned} \parallel \sum_{l=1}^q m_f(r, D_l) & \leq -\frac{n+1}{mH_Y(m)} N_{C(F)}(r, 0) + \frac{\varepsilon}{3m} T_F(r) + d(n+1)T_f(r) \\ & + \frac{(2n+1)(n+1)\Delta}{m} \sum_{1 \leq l \leq q} m_f(r, D_l) + S(r, F). \end{aligned}$$

Next, we estimate

$$\sum_{l=1}^q N_f(r, D_l) - \frac{n+1}{mH_Y(m)} N_{C(F)}(r, 0).$$

For  $z \in \mathbb{C}$ , we may assume that  $Q_j \circ f(\zeta + lc)$  vanishes at  $z$  for  $0 \leq j \leq q_1$  and  $Q_j \circ f(\zeta + lc)$  does not vanish at  $z$  for  $j > q_1, l = 1, \dots, n_m$ . By in general position assumption, we know that  $q_1 \leq n$ . There are integers  $k_j \geq 0$  and nowhere vanishing holomorphic functions  $g_{jl}$  in a neighborhood  $U$  of  $z$  such that

$$Q_j \circ f(\zeta + lc) = (\zeta - z)^{k_j} g_{jl}, j = 1, \dots, q,$$

where  $k_j > 0, j = 1, \dots, q_1$  and  $k_j = 0, q_1 < j \leq q$ . Let  $c'_z = (k_1, \dots, k_{q_1}, 0, \dots, 0)$ . There is a subset  $I'_z \subset \{0, 1, \dots, q_m\}$  with  $|I'_z| = m + 1 = H_Y(m)$  such that  $\{y^{a_i} : i \in I'_z\}$  is a basis of  $\mathbb{C}[y_1, \dots, y_q]_m / (I_Y)_m$  and

$$S_Y(m, c'_z) = \sum_{i \in I'_z} a_i \cdot c'_z.$$

Thus, from  $y^{a_i} = L_{j,z}(\phi_0, \dots, \phi_{n_m})$ , we can obtain linearly independent linear forms  $L_i, i \in I'_z$ . Hence, we get

$$L_i(F(\zeta + lc)) = (Q_1(f(\zeta + lc)))^{a_{i,1}} \dots (Q_q(f(\zeta + lc)))^{a_{i,q}}, i \in I'_z.$$

where  $(Q_j(f(\zeta + lc)))^{a_{i,j}} = (\zeta - z)^{a_{i,j}k_j} g_{jl}^{a_{i,j}}$  for all  $i \leq j \leq q$ . We can assume that  $I'_z = \{0, 1, \dots, n_m\}$ . By the property of Casorati determinant and Lemma 2, we have

$$C(F_0, \dots, F_{n_m}) = CC(L_0(F), \dots, L_{n_m}(F)) \neq 0,$$

where  $C$  is a nonzero constant. Thus  $C(F_0, \dots, F_{n_m})$  vanishes at  $z$  with order at least

$$\sum_{i \in I'_z, 1 \leq j \leq n_m} a_{i,j} k_j = \sum_{i \in I'_z} a_i c'_z.$$

By Lemma 3, we have

$$S_Y(m, c'_z) \geq \frac{mH_Y(m)}{n+1} \sum_{1 \leq j \leq q_1} k_j - (2n+1)\Delta H_Y(m) \max_{1 \leq j \leq q_1} k_j.$$

Therefore,  $C(F_0, \dots, F_{n_m})$  vanishes at  $z$  with multiple at least

$$\frac{mH_Y(m)}{n+1} \sum_{1 \leq j \leq q_1} k_j - (2n+1)\Delta H_Y(m) \max_{1 \leq j \leq q_1} k_j.$$

By definition of  $N_f(r, D_j, )$ ,  $N_{C(F)}(r, 0)$  and  $\tilde{N}_f^{[n_m, c]}(r, D)$ , we have

$$\begin{aligned} \frac{mH_Y(m)}{n+1} \sum_{j=1}^q N_f(r, D_j) - N_{C(F)}(r, 0) &\leq \frac{mH_Y(m)}{n+1} \sum_{j=1}^q \tilde{N}_f^{[n_m, c]}(r, D_j) \\ (3.12) \quad &+ (2n+1)\Delta H_Y(m) \sum_{j=1}^q N_f(r, D_j). \end{aligned}$$

By First main theorem, we see  $T_F(r) \leq dmT_f(r) + O(1)$ . Thus, (3.11) and (3.12) imply

$$\begin{aligned} \parallel \sum_{l=1}^q d(q - (n+1) - \varepsilon/3)T_f(r) &\leq \sum_{l=1}^q \tilde{N}_f^{[n_m, c]}(r, D_l) \\ (3.13) \quad &+ \frac{(2n+1)(n+1)\Delta}{m} qdT_f(r) + S(r, f). \end{aligned}$$

We choose the  $m$  sufficiently large such that

$$(3.14) \quad \frac{(2n+1)(n+1)\Delta}{m} < \varepsilon/3.$$

We may choose  $m = 18n^2\Delta I(\varepsilon^{-1})$  for the inequality (3.14), where  $I(x) := \min\{k \in \mathbb{N} : k > x\}$  for each positive constant  $x$ . Thus, from (3.13) and (3.14), we get the inequality

$$\parallel d(q(1 - \varepsilon/3) - (n+1) - \varepsilon/3)T_f(r) \leq \sum_{l=1}^q \tilde{N}_f^{[n_m, c]}(r, D_l) + S(r, f).$$

By property  $\deg Y = \Delta \leq d^n \deg V$ , where  $d = \text{lcm}\{d_1, \dots, d_q\}$ ,  $\deg Y = n$  and  $n_m \leq \Delta C_{m+n}^n$ , we have

$$\begin{aligned} n_m &\leq \Delta \frac{(m+1)(m+2)\dots(m+n)}{n!} \\ &< \Delta \left(\frac{m+n}{n}\right)^n \frac{n^n}{n!} \\ &= \Delta \left(1 + \frac{m}{n}\right)^n \frac{n^n}{n!}. \end{aligned}$$

For the choice of  $m$ , we have

$$n_m \leq \frac{n^n d^{n^2+n} (19nI(\varepsilon^{-1}))^n (\deg V)^{n+1}}{n!}.$$

□

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